ANALYSE VECTORIELLE

Soient U et V deux champs scalaires et $\vec{a}$ et $\vec{b}$ deux champs vectoriels.

Attention l'équivalence des formules avec celles données avec la notation $\vec{\nabla}$ n'est rigoureuse qu'avec le système de coordonnées cartésiennes. Dans les autres cas, c'est juste une facilité de notation tolérée.

1.1 Formules portant sur un seul champ

1. $\boxed{\vec{\nabla}.(\vec{\nabla}U) = \vec{\nabla}^2 U \text{ soit } div(\overrightarrow{grad}\,U) = \Delta U}$
2. $\boxed{\vec{\nabla} \wedge (\vec{\nabla}U) = \vec{0} \text{ soit } \overrightarrow{rot}(\overrightarrow{grad}\,U) = \vec{0}}$
3. $\boxed{\vec{\nabla}.(\vec{\nabla} \wedge \vec{a}) = \vec{0} \text{ soit } div(\overrightarrow{rot}\,\vec{a}) = \vec{0}}$
4. $\boxed{\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{a}) = \vec{\nabla}(\vec{\nabla}.\vec{a}) - \vec{\nabla}^2\vec{a} \text{ soit } \overrightarrow{rot}(\overrightarrow{rot}\,\vec{a}) = \overrightarrow{grad}(div\vec{a}) - \Delta\vec{a}}$

1.2 Formules portant sur deux champs

1. $\vec{\nabla}(UV) = V\vec{\nabla}(U) + U\vec{\nabla}(V)$ soit $\overrightarrow{grad}(UV) = V\,\overrightarrow{grad}\,U + U\,\overrightarrow{grad}\,V$
2. $\boxed{\vec{\nabla}.(U\vec{a}) = \vec{a}(\vec{\nabla}U) + U(\vec{\nabla}.\vec{a}) \text{ soit } div(U\vec{a}) = \overrightarrow{grad}\,U.\vec{a} + U div\vec{a}}$
3. $\boxed{\vec{\nabla} \wedge (U\vec{a}) = (\vec{\nabla}U) \wedge \vec{a} + U(\vec{\nabla} \wedge \vec{a}) \text{ soit } \overrightarrow{rot}(U\vec{a}) = \overrightarrow{grad}\,U \wedge \vec{a} + U\,\overrightarrow{rot}\,\vec{a}}$
4. $\vec{\nabla}.(\vec{a} \wedge \vec{b}) = \vec{b}.(\vec{\nabla} \wedge \vec{a}) - \vec{a}.(\vec{\nabla} \wedge \vec{b})$ soit $div(\vec{a} \wedge \vec{b}) = \vec{b}.\overrightarrow{rot}\,\vec{a} - \vec{a}.\overrightarrow{rot}\,\vec{b}$
5. $\vec{\nabla} \wedge (\vec{a} \wedge \vec{b}) = (\vec{\nabla}.\vec{b})\vec{a} - (\vec{\nabla}.\vec{a})\vec{b} + (\vec{b}.\vec{\nabla})\vec{a} - (\vec{a}.\vec{\nabla})\vec{b}$
   soit $\overrightarrow{rot}(\vec{a} \wedge \vec{b}) = (div\vec{b})\vec{a} - (div\vec{a})\vec{b} + (\vec{b}.\overrightarrow{grad})\vec{a} - (\vec{a}.\overrightarrow{grad})\vec{b}$
6. $\vec{\nabla}(\vec{a}.\vec{b}) = \vec{a} \wedge (\vec{\nabla} \wedge \vec{b}) + \vec{b} \wedge (\vec{\nabla} \wedge \vec{a}) + (\vec{b}.\vec{\nabla})\vec{a} + (\vec{a}.\vec{\nabla})\vec{b}$
   soit $\overrightarrow{grad}(\vec{a}.\vec{b}) = \vec{a} \wedge (\overrightarrow{rot}\,\vec{b}) + \vec{b} \wedge (\overrightarrow{rot}\,\vec{a}) + (\vec{b}.\overrightarrow{grad})\vec{a} + (\vec{a}.\overrightarrow{grad})\vec{b}$

1.3 Expressions des opérateurs dans divers systèmes de coordonnées

1.3.1 Gradient

* cartésiennes : $\boxed{\overrightarrow{grad}\,U = \left(\frac{\partial U}{\partial x}\right)\vec{e}_x + \left(\frac{\partial U}{\partial y}\right)\vec{e}_y + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z}$
* cylindriques : $\boxed{\overrightarrow{grad}\,U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \left(\frac{\partial U}{\partial z}\right)\vec{e}_z}$
* sphériques : $\overrightarrow{grad}\,U = \left(\frac{\partial U}{\partial r}\right)\vec{e}_r + \frac{1}{r}\left(\frac{\partial U}{\partial \theta}\right)\vec{e}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial U}{\partial \varphi}\right)\vec{e}_\varphi$

1.3.2 Divergence

* cartésiennes : $\boxed{div\vec{a} = \left(\frac{\partial a_x}{\partial x}\right) + \left(\frac{\partial a_y}{\partial y}\right) + \left(\frac{\partial a_z}{\partial z}\right)}$
* cylindriques : $div\vec{a} = \frac{1}{r}\left(\frac{\partial r a_r}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right) + \left(\frac{\partial a_z}{\partial z}\right)$. $\boxed{\text{Si } \vec{a} = a_r(r)\vec{e_r} \text{ alors } div\vec{a} = \frac{1}{r}\left(\frac{\partial r a_r}{\partial r}\right)}$
* sphériques : $div\vec{a} = \frac{1}{r^2}\left(\frac{\partial r^2 a_r}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial a_\theta \sin\theta}{\partial \theta}\right) + \frac{1}{r\sin\theta}\left(\frac{\partial a_\varphi}{\partial \varphi}\right)$. $\boxed{\text{Si } \vec{a} = a_r(r)\vec{e_r} \text{ alors } div\vec{a} = \frac{1}{r^2}\left(\frac{\partial r^2 a_r}{\partial r}\right)}$

1.3.3 Rotationnel

* cartésiennes : $\boxed{\vec{rot}\,\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z}$

* cylindriques : $\vec{rot}\,\vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_z$

* sphériques : $\vec{rot}\,\vec{a} = \frac{1}{r\sin\theta}\left(\frac{\partial (a_\varphi \sin\theta)}{\partial \theta} - \frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial (r a_\varphi)}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial (r a_\theta)}{\partial r} - \frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi$

1.3.4 Laplacien

* cartésiennes : $\boxed{\Delta U = \left(\frac{\partial^2 U}{\partial x^2}\right) + \left(\frac{\partial^2 U}{\partial y^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)}$

* cylindriques : $\Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 U}{\partial \theta^2}\right) + \left(\frac{\partial^2 U}{\partial z^2}\right)$. $\boxed{\text{Si } U = U(r) \text{ alors } \Delta U = \frac{1}{r}\left(\frac{\partial}{\partial r} r\left(\frac{\partial U}{\partial r}\right)\right)}$

* sphériques : $\Delta U = \left(\frac{\partial^2 U}{\partial r^2}\right) + \frac{2}{r}\left(\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 U}{\partial \varphi^2}\right) + \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right)\right)$.

$$\boxed{\text{Si } U = U(r) \text{ alors } \Delta U = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2\left(\frac{\partial U}{\partial r}\right)\right)}$$

1.4 Action des opérateurs sur le trièdre de base

1.4.1 Coordonnées cylindriques

Rotationnel

$$\boxed{\vec{rot}\vec{e}_r = \vec{0} \quad \vec{rot}\vec{e}_\theta = \frac{\vec{e}_z}{r} \quad \vec{rot}\vec{e}_z = \vec{0}}$$

Divergence

$$\boxed{div\vec{e}_r = \frac{1}{r} \quad div\vec{e}_\theta = 0 \quad div\vec{e}_z = 0}$$

1.4.2 Coordonnées sphériques

Rotationnel

$$\boxed{\vec{rot}\vec{e}_r = \vec{0} \quad \vec{rot}\vec{e}_\theta = \frac{\vec{e}_\varphi}{r} \quad \vec{rot}\vec{e}_\varphi = \frac{\cos\theta}{r\sin\theta}\vec{e}_r - \frac{\vec{e}_\theta}{r}}$$

Divergence

$$\boxed{div\vec{e}_r = \frac{2}{r} \quad div\vec{e}_\theta = \frac{1}{r\tan\theta} \quad div\vec{e}_\varphi = 0}$$